Regularity estimation in multivariate functional data analysis

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$$H = \frac{\log\left(\mathbb{E}\left[\left\{B^H(t) - B^H(s)\right\}^2\right]\right)}{2\log|t-s|}$$

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  for any t and s in a neighbourhood of t_0

- $$H(t_0) \approx \frac{\log(\theta(t_1, t_2)) - \log(\theta(t_1, t_3))}{2\log(2)},$$

  where

$$\theta(t,s) = \mathbb{E}\left[\{X(t) - X(s)\}^2\right] \quad \text{and} \quad |t_1 - t_2| = 2|t_1 - t_3|.$$

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- The data associated to a sample path $X^{(i)}$ consist of the pairs $(Y_m^{(i)}, \boldsymbol{t}_m^{(i)}) \in \mathbb{R} \times \mathcal{T}$ where $Y_m^{(i)}$ is defined as

$$Y_m^{(i)} = X^{(i)}(\boldsymbol{t}_m^{(i)}) + \varepsilon_m^{(i)}, \quad \text{with} \quad \varepsilon_m^{(i)} = \sigma(\boldsymbol{t}_m^{(i)}, X(\boldsymbol{t}_m^{(i)}))e_m^{(i)}$$

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- $M_1, \ldots, M_N$ be an independent sample of an integer-valued random variable M, $\mathbb{E}[M] = \mathfrak{m}$
- The $\left(\boldsymbol{t}_m^{(i)}, \quad 1 \le m \le M_i\right)$ represent the observation points for the sample path $X^{(i)}$.

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- For $X \in L^2$, we denote for sufficiently small scalars Δ
$$\theta_{\boldsymbol{t}}^{(i)}(\Delta) = \mathbb{E}\left[\left\{X\left(\boldsymbol{t} - \frac{\Delta}{2}e_i\right) - X\left(\boldsymbol{t} + \frac{\Delta}{2}e_i\right)\right\}^2\right], \quad i = 1, 2,$$
where (e_1, e_2) is canonical basis of $\mathbb{R}^2$

A class of multivariate processes

Definition

$X \in \mathcal{H}^{H_1,H_2}(\boldsymbol{L},\mathcal{T})$ if three constants $\Delta_0, C, \beta > 0$ exist such that for any $\boldsymbol{t} \in \mathcal{T}$ and $0 < \Delta \leq \Delta_0$,

$$\left|\theta_{\boldsymbol{t}}^{(i)}(\Delta) - L_1^{(i)}(\boldsymbol{t})\Delta^{2H_1(\boldsymbol{t})} - L_2^{(i)}(\boldsymbol{t})\Delta^{2H_2(\boldsymbol{t})}\right| \leq C\Delta^{2\overline{H}(\boldsymbol{t})+\beta}, \quad i = 1, 2.$$

Let

$$\mathcal{H}^{H_1,H_2} = \mathcal{H}^{H_1,H_2}(\mathcal{T}) = \bigcup_{\boldsymbol{L}} \mathcal{H}^{H_1,H_2}(\boldsymbol{L},\mathcal{T}),$$

where $\boldsymbol{L} = (L_1^{(1)}, L_2^{(1)}, L_1^{(2)}, L_2^{(2)})$. The functions H_1, H_2 define the local regularity of the process, while $\boldsymbol{L}$ represent the local Hölder constants.

Identification

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- Assume $X \in \mathcal{H}^{H_1,H_2}$ and $X \in \mathcal{H}^{\tilde{H}_1,\tilde{H}_2}$
- We then necessarily have

$$\min\{H_1(\boldsymbol{t}), H_2(\boldsymbol{t})\} = \min\{\tilde{H}_1(\boldsymbol{t}), \tilde{H}_2(\boldsymbol{t})\},$$

and

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- Notation :

$$\underline{H}(\boldsymbol{t}) = \min\{H_1(\boldsymbol{t}), H_2(\boldsymbol{t})\}, \quad \overline{H}(\boldsymbol{t}) = \max\{H_1(\boldsymbol{t}), H_2(\boldsymbol{t})\}.$$

Estimating equations for $\underline{H}$ and $\overline{H}$

Denote for any $\boldsymbol{t} \in \mathcal{T}$

$$\gamma_{\boldsymbol{t}}(\Delta) = \theta_{\boldsymbol{t}}^{(1)}(\Delta) + \theta_{\boldsymbol{t}}^{(2)}(\Delta)$$

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Let

$$\alpha_{\boldsymbol{t}}(\Delta) = \left| \frac{\gamma_{\boldsymbol{t}}(2\Delta)}{(2\Delta)^{2\underline{H}(\boldsymbol{t})}} - \frac{\gamma_{\boldsymbol{t}}(\Delta)}{\Delta^{2\underline{H}(\boldsymbol{t})}} \right|.$$

$$\overline{H}(\boldsymbol{t}) - \underline{H}(\boldsymbol{t}) \approx \frac{\log(\alpha_{\boldsymbol{t}}(2\Delta)) - \log(\alpha_{\boldsymbol{t}}(\Delta))}{2\log(2)}.$$

- In general, the sheets $X^{(j)}$, $j \in \{1, \ldots, N\}$, are not available
- Let $\widetilde{X}^{(j)}$ be an observable approximation of $X^{(j)}$.
  - If X is observed everywhere and without noise, then

$$\widetilde{X}^{(j)}(\boldsymbol{t}) = X^{(j)}(\boldsymbol{t}), \quad \forall \boldsymbol{t} \in \mathcal{T}$$

  - If X is observed with noise or/and on a discrete grid, then $\widetilde{X}^{(j)}$ is an estimator of $X^{(j)}$ (local polynomial, splines, interpolation...)

- For $i = 1, 2$. $\theta_{\boldsymbol{t}}^{(i)}(\Delta)$ can be estimated by :

$$\widehat{\theta_{\boldsymbol{t}}^{(i)}}(\Delta) = \frac{1}{N} \sum_{j=1}^{N} \left\{ \widetilde{X}^{(j)}(\boldsymbol{t} - (\Delta/2)e_i) - \widetilde{X}^{(j)}(\boldsymbol{t} + (\Delta/2)e_i) \right\}^2,$$

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we obtain an estimator of $\underline{H}(\boldsymbol{t})$:

$$\widehat{\underline{H}}(\boldsymbol{t}) = \begin{cases} \frac{\log(\widehat{\gamma}_{\boldsymbol{t}}(2\Delta)) - \log(\widehat{\gamma}_{\boldsymbol{t}}(\Delta))}{2\log(2)} & \text{if} \quad \widehat{\gamma}_{\boldsymbol{t}}(2\Delta), \widehat{\gamma}_{\boldsymbol{t}}(\Delta) > 0 \\ 1 & \text{otherwise} \end{cases}.$$

Moreover

$$\widehat{\alpha}_{\boldsymbol{t}}(\Delta) = \begin{cases} \left| \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(2\Delta)}{(2\Delta)^{2\underline{\widehat{H}}(\boldsymbol{t})}} - \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(\Delta)}{\Delta^{2\underline{\widehat{H}}(\boldsymbol{t})}} \right| & \text{if} \quad \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(2\Delta)}{(2\Delta)^{2\underline{\widehat{H}}(\boldsymbol{t})}} \neq \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(\Delta)}{\Delta^{2\underline{\widehat{H}}(\boldsymbol{t})}} \\ 1 & \text{otherwise.} \end{cases}.$$

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Hence

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$$(\widehat{\overline{H} - \underline{H}})(\boldsymbol{t}) = \frac{\log(\widehat{\alpha}_{\boldsymbol{t}}(2\Delta)) - \log(\widehat{\alpha}_{\boldsymbol{t}}(\Delta))}{2\log(2)}.$$

We set

$$A_N(\tau) = \left\{ (\widehat{\overline{H} - \underline{H}})(\boldsymbol{t}) \geq \tau \right\},$$

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and define

$$\widehat{\overline{H}}(\boldsymbol{t}) = \widehat{\underline{H}}(\boldsymbol{t}) + (\widehat{\overline{H} - \underline{H}})(\boldsymbol{t}) 1_{A_N(\tau)}.$$

Assumption

- The observable approximation of $X^{(j)}$ is such that

$$\mathbb{P}\left(\widehat{\theta_{\boldsymbol{t}}^{(i)}}(\Delta) - \theta_{\boldsymbol{t}}^{(i)}(\Delta) \geq \varepsilon\right) \leq \exp\left(-\mathfrak{u}N\varepsilon^2\varrho(\Delta)\right),$$

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  - If X is observed in a random grid and with noise,

$$\varrho(\Delta) = 1$$

Concentration bounds (1/3)

There exist five constants $L_1, \dots, L_5$ such that $\forall \varepsilon \in (0,1)$

$$\mathbb{P}\left[|\widehat{\underline{H}}(\boldsymbol{t}) - \underline{H}(\boldsymbol{t})| \geq \varepsilon\right] \leq L_1 \exp\left(-L_2 N \varepsilon^2 \Delta^{4\underline{H}(\boldsymbol{t})} \varrho(\Delta)\right)$$

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and

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where

$$p_1 = \exp\left[-L_4 N \tau^2 \frac{\Delta^{4\overline{H}(\boldsymbol{t})} \varrho(\Delta)}{\log^2(\Delta)} \Delta^{4D(\boldsymbol{t})}\right],$$

$$p_2 = \exp\left[-L_5 N \varepsilon^2 \frac{\Delta^{4\overline{H}(\boldsymbol{t})} \varrho(\Delta)}{\log^2(\Delta)} \Delta^{4D(\boldsymbol{t})}\right] 1_{\underline{H}(\boldsymbol{t}) < \overline{H}(\boldsymbol{t})}.$$

Concentration bounds (2/3)

Constants $C_1, \ldots, C_4$ exist such $\forall \varepsilon \in (0,1)$ and $i = 1, 2$:

$$\mathbb{P}\left(\left|\widehat{L_1^{(i)}}(\boldsymbol{t}) - L_1^{(i)}(\boldsymbol{t})\right| \geq \varepsilon\right) \leq C_1 \exp\left(-C_2 N \varepsilon^2 \frac{\Delta^{4\underline{H}(\boldsymbol{t})}\varrho(\Delta)}{\log^2(\Delta)}\right)$$

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and

$$\mathfrak{G}_\varepsilon^{(i)} \leq C_3 \exp\left(-C_4 N \varepsilon \min\{\varepsilon, \Delta^{4D(\boldsymbol{t})}\}(2^{2D(\boldsymbol{t})} - 1)^2 \frac{\Delta^{4\overline{H}(\boldsymbol{t})} \varrho(\Delta)}{\log^4(\Delta)} \Delta^{4D(\boldsymbol{t})}\right)$$

where

$$\mathfrak{G}_\varepsilon^{(i)} = \mathbb{P}\left(\left|\widehat{L_2^{(i)}}(\boldsymbol{t}) - L_2^{(i)}(\boldsymbol{t})\right| \geq \varepsilon\right).$$

Concentration bounds (3/3)

- Let

$$\tau \leq \left\{ \overline{H}(\boldsymbol{t}) - \underline{H}(\boldsymbol{t}) \right\} /2 + 1_{\{\underline{H}(\boldsymbol{t}) = \overline{H}(\boldsymbol{t})\}},$$

and

$$\Delta = \exp(-\log^{\varrho}(1/\tau)),$$

for some $\varrho \in (0, 1)$.

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- Then
$$\mathbb{P}\left(1_{A_N(\tau)} \neq 1_{\{\underline{H}(\boldsymbol{t})\neq\overline{H}(\boldsymbol{t})\}}\right) \leq \exp\left[-L_4 N\tau^2 \frac{\Delta^{4\overline{H}(\boldsymbol{t})}\varrho(\Delta)}{\log^2(\Delta)}\Delta^{4D(\boldsymbol{t})}\right].$$

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- Estimation of the nonparametric characteristics of the MfBs.

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- Assume that $X \in \mathcal{H}^{H_1,H_2}(L_1, 0, 0, L_2)$
- A new realisation is observed X^{new}.
- We propose a smoother that is optimal.

Take home message

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- Nonparametric estimator for the regularity was introduced.
- The estimator proposed adapt to the isotropic case.
- Knowing the regularity helps to construct optimal estimation procedures.

THANK YOU

References

- Ayache, A., S. Cohen, and J. Lévy Véhel (2000). The covariance structure of multifractional Brownian motion, with application to long range dependence. In 2000 IEEE International Conference on Acoustics, Speech, and Signal Processing - ICASSP 2000, Istanbul, Turkey.
- Golovkine, S., N. Klutchnikoff, and V. Patilea (2022). Learning the smoothness of noisy curves with application to online curve estimation. Electron. J. Stat. 16 (1), 1485–1560.
- Hoffmann, M. and O. Lepski (2002). Random rates in anisotropic regression. (With discus- sion). Ann. Stat. 30(2), 325–396.
- Peltier, R.-F. and J. Lévy Véhel (1995). Multifractional Brownian Motion : Definition and Preliminary Results. Research Report RR-2645, INRIA. Projet FRACTALES.
- Stoev, S. A. and M. S. Taqqu (2006). How rich is the class of multifractional Brownian motions ? Stochastic Processes Appl. 116(2), 200–221.