Regularity estimation in multivariate functional data analysis

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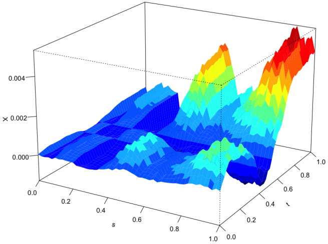
Multivariate functional data

- The realizations of the stochastic process X are surfaces
    - Satellite images
    - Measurements of temperature or salinity in oceanology



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- The data associated to a sample path $X^{(j)}$ consist of the pairs $(Y_m^{(j)}, \boldsymbol{t}_m^{(j)}) \in \mathbb{R} \times \mathcal{T}$, where for $1 \leq j \leq N$ and $1 \leq m \leq M_j$

$$Y_m^{(j)} = X^{(j)}(\boldsymbol{t}_m^{(j)}) + \varepsilon_m^{(j)}, \quad \text{with} \quad \varepsilon_m^{(j)} = \sigma(\boldsymbol{t}_m^{(j)}, X^{(j)}(\boldsymbol{t}_m^{(j)}))e_m^{(j)}$$

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- $M_1, \ldots, M_N$ be an independent sample of an integer-valued random variable M, $\mathbb{E}[M] = \mathfrak{m}$
- The $\left(\boldsymbol{t}_m^{(j)}, \quad 1 \leq m \leq M_j\right)$ represent the observation points for the sample path $X^{(j)}$.

First steps : univariate case (1/2)

- For B^H a fBm with Hurst index $H \in (0,1)$,

$$\mathbb{E}\left[\left\{B^H(t) - B^H(s)\right\}^2\right] = |t-s|^{2H}, \quad s,t \in \mathbb{R}_+$$

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- Estimating equation for the Hurst parameter :

$$H = \frac{\log\left(\mathbb{E}\left[\left\{B^H(t) - B^H(s)\right\}^2\right]\right)}{2\log|t-s|}$$

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  for t and s close to t_0
- Estimating equation :

$$H(t_0) \approx \frac{\log(\theta(t_1,t_2)) - \log(\theta(t_1,t_3))}{2\log(2)}, \qquad t_0 \in [t_1,t_2] \subset [t_1,t_3]$$

  where

$$\theta(t,s) = \mathbb{E}\left[\{X(t) - X(s)\}^2\right] \quad \text{and} \quad |t_1 - t_2| = 2|t_1 - t_3|.$$

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- For $X \in \mathcal{L}^2$, we denote for sufficiently small scalars Δ

$$\theta_{\boldsymbol{t}}^{(i)}(\Delta) = \mathbb{E}\left[\left\{X\left(\boldsymbol{t} + \frac{\Delta}{2}e_i\right) - X\left(\boldsymbol{t} - \frac{\Delta}{2}e_i\right)\right\}^2\right], \quad i = 1,2,$$

where (e_1, e_2) is canonical basis of $\mathbb{R}^2$

A class of multivariate processes

Definition

We say $X \in \mathcal{H}^{H_1,H_2}(\boldsymbol{L},\mathcal{T})$ if three constants $\Delta_0, C, \beta > 0$ exist such that for any $\boldsymbol{t} \in \mathcal{T}$ and $0 < \Delta \leq \Delta_0$,

$$\left|\theta_{\boldsymbol{t}}^{(i)}(\Delta) - L_1^{(i)}(\boldsymbol{t})\Delta^{2H_1(\boldsymbol{t})} - L_2^{(i)}(\boldsymbol{t})\Delta^{2H_2(\boldsymbol{t})}\right| \leq C\Delta^{2\overline{H}(\boldsymbol{t})+\beta}, \quad i = 1, 2.$$

Let

$$\mathcal{H}^{H_1,H_2} = \bigcup_{\boldsymbol{L}} \mathcal{H}^{H_1,H_2}(\boldsymbol{L},\mathcal{T}),$$

where $\boldsymbol{L} = (L_1^{(1)}, L_2^{(1)}, L_1^{(2)}, L_2^{(2)})$.

The functions H_1, H_2 define the local regularity of the process, while $\boldsymbol{L}$ represent the local Hölder constants.

Example : Sum of two fractional Brownian motion

- Let $B_1^{H_1}$ and $B_2^{H_2}$ be two independent fBm with Hurst index H_1 and H_2.
- Let

$$X_1(\boldsymbol{t}) = B_1^{H_1}(t_1) + B_2^{H_2}(t_2), \quad \forall \boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2.$$

  Then $X_1 \in \mathcal{H}^{H_1,H_2}$ where $\boldsymbol{L} = (1, 0, 0, 1)$.
- Let $\beta > 0$ and define

$$X_2(\boldsymbol{t}) = X_1\left(\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \boldsymbol{t}\right), \qquad \forall \boldsymbol{t} \in \mathbb{R}^2.$$

  Then $X_2 \in \mathcal{H}^{H_1,H_2}$ with

$$\boldsymbol{L} = (|\cos\beta|^{2H_1}, |\sin\beta|^{2H_2}, |\sin\beta|^{2H_1}, |\cos\beta|^{2H_2}).$$

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- We then necessarily have

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- Notation :

$$\underline{H}(\boldsymbol{t}) = \min\{H_1(\boldsymbol{t}), H_2(\boldsymbol{t})\}, \quad \overline{H}(\boldsymbol{t}) = \max\{H_1(\boldsymbol{t}), H_2(\boldsymbol{t})\}.$$

Estimating equations for $\underline{H}$ and $\overline{H}$

- Recall

$$\theta_{\boldsymbol{t}}^{(i)}(\Delta) = \mathbb{E}\left[\left\{X\left(\boldsymbol{t} - \Delta e_i/2\right) - X\left(\boldsymbol{t} + \Delta e_i/2\right)\right\}^2\right], \quad i = 1, 2,$$

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  - $$\sup_{s \in \mathcal{T}} \mathbb{E}\left[(X(s) - \widetilde{X}(s))^{2p}\right] \leq C_p \rho(\mathfrak{m})^{2p}$$

Estimators for $\underline{H}$ and $\overline{H}$ (1/2)

- The observable approximation allows to build estimates :

$$\widehat{\theta}_{\boldsymbol{t}}^{(i)}(\Delta) = \frac{1}{N}\sum_{j=1}^{N}\left\{\widetilde{X}^{(j)}(\boldsymbol{t} - (\Delta/2)e_i) - \widetilde{X}^{(j)}(\boldsymbol{t} + (\Delta/2)e_i)\right\}^2,$$

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- The first estimator follows :

$$\widehat{\underline{H}}(\boldsymbol{t}) = \begin{cases} \frac{\log(\widehat{\gamma}_{\boldsymbol{t}}(2\Delta)) - \log(\widehat{\gamma}_{\boldsymbol{t}}(\Delta))}{2\log(2)} & \text{if} \quad \widehat{\gamma}_{\boldsymbol{t}}(2\Delta), \widehat{\gamma}_{\boldsymbol{t}}(\Delta) > 0 \\ 1 & \text{otherwise} \end{cases}.$$

Estimators for $\underline{H}$ and $\overline{H}$ (2/2)

- Moreover

$$\widehat{\alpha}_{\boldsymbol{t}}(\Delta) = \begin{cases} \left| \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(2\Delta)}{(2\Delta)^{2\widehat{\underline{H}}(\boldsymbol{t})}} - \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(\Delta)}{\Delta^{2\widehat{\underline{H}}(\boldsymbol{t})}} \right| & \text{if} \quad \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(2\Delta)}{(2\Delta)^{2\widehat{\underline{H}}(\boldsymbol{t})}} \neq \dfrac{\widehat{\gamma}_{\boldsymbol{t}}(\Delta)}{\Delta^{2\widehat{\underline{H}}(\boldsymbol{t})}} \\ 1 & \text{otherwise} \end{cases}.$$

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- We then set

$$A_N(\tau) = \left\{ \widehat{(\overline{H} - \underline{H})}(\boldsymbol{t}) \geq \tau \right\},$$

and define

$$\widehat{\overline{H}}(\boldsymbol{t}) = \widehat{\underline{H}}(\boldsymbol{t}) + \widehat{(\overline{H} - \underline{H})}(\boldsymbol{t}) 1_{A_N(\tau)}.$$

Estimating equations for $L_1^{(i)}(\boldsymbol{t})$ and $L_2^{(i)}(\boldsymbol{t})$

- Recall

$$\theta_{\boldsymbol{t}}^{(i)}(\Delta) \approx L_1^{(i)}(\boldsymbol{t})\Delta^{2H_1(\boldsymbol{t})} + L_2^{(i)}(\boldsymbol{t})\Delta^{2H_2(\boldsymbol{t})}, \quad i = 1, 2$$

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- For $i = 1, 2$,

$$L_1^{(i)}(\boldsymbol{t}) \approx \frac{\theta_{\boldsymbol{t}}^{(i)}(\Delta)}{\Delta^{2H_1(\boldsymbol{t})}}$$

and

$$L_2^{(i)}(\boldsymbol{t}) \approx \frac{1}{(4^{D(\boldsymbol{t})} - 1)\Delta^{2D(\boldsymbol{t})}} \left| \frac{\theta_{\boldsymbol{t}}^{(i)}(2\Delta)}{(2\Delta)^{2H_1(\boldsymbol{t})}} - \frac{\theta_{\boldsymbol{t}}^{(i)}(\Delta)}{\Delta^{2H_1(\boldsymbol{t})}} \right|$$

with $D(\boldsymbol{t}) = H_2(\boldsymbol{t}) - H_1(\boldsymbol{t})$

Estimators for $L_1^{(i)}(\boldsymbol{t})$ and $L_2^{(i)}(\boldsymbol{t})$

- Plug into the estimating equations for $L_j^{(i)}(\boldsymbol{t})$ the estimators of the unknown quantities, as defined above
- Special attention requires the case $\underline{H}(\boldsymbol{t}) = \overline{H}(\boldsymbol{t})$
  - A diagnostic tool is provided

Non-asymptotic results

Proposition 1 : Constants $C_1, \ldots, C_5$ exist such that,

$$\forall \varepsilon, \tau \in (0,1) \quad \max\{|\log(\Delta)||R(\underline{H})(\boldsymbol{t})|,\ |R(\overline{H}-\underline{H})(\boldsymbol{t})|\} \leq \varepsilon \leq 2\tau,$$

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with

- $p_1 = C_1 \exp\left(-C_2 N \times \varepsilon^2 \times \Delta^{4\underline{H}(\boldsymbol{t})} \varrho(\Delta, \mathfrak{m})\right)$,

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$$\mathbb{P}\left[\left|\widehat{\overline{H}}(\boldsymbol{t}) - \overline{H}(\boldsymbol{t})\right| \geq \varepsilon\right] \leq C_3\{p_1 + p_2 + p_3\},$$

with

- $p_1 = C_1 \exp\left(-C_2 N \times \varepsilon^2 \times \Delta^{4\underline{H}(\boldsymbol{t})} \varrho(\Delta, \mathfrak{m})\right)$,
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Non-asymptotic results

Proposition 1 : Constants $C_1, \ldots, C_5$ exist such that,

$$\forall \varepsilon, \tau \in (0,1) \quad \max\{|\log(\Delta)||R(\underline{H})(\boldsymbol{t})|,\ |R(\overline{H}-\underline{H})(\boldsymbol{t})|\} \leq \varepsilon \leq 2\tau,$$

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where $\varrho(\Delta, \mathfrak{m}) = \max\{\Delta^{2\underline{H}(\boldsymbol{t})}, \rho(\mathfrak{m})^2\}^{-1}.$

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Proposition 3 : Let

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where C_3 and C_5 are the positive constants from Proposition 1.

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- Then for $\boldsymbol{t}, \boldsymbol{s} \in \mathcal{T}$, we have

$$\begin{aligned}\theta(\boldsymbol{t}, \boldsymbol{s}) &= \mathbb{E}\left[\{X(\boldsymbol{t}) - X(\boldsymbol{s})\}^2\right] \\ &\approx |A_1(\boldsymbol{t})|^{2H_1(\boldsymbol{t})}|\partial_1 A_2(\boldsymbol{t})(t_1 - s_1) + \partial_2 A_2(\boldsymbol{t})(t_2 - s_2)|^{2H_2(\boldsymbol{t})} \\ &\quad + |A_2(\boldsymbol{t})|^{2H_2(\boldsymbol{t})}|\partial_1 A_1(\boldsymbol{t})(t_1 - s_1) + \partial_2 A_1(\boldsymbol{t})(t_2 - s_2)|^{2H_1(\boldsymbol{t})},\end{aligned}$$

where

$$H_1 = \eta_1 \circ A \quad \text{and} \quad H_2 = \eta_2 \circ A.$$

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  - Multifractional Brownian sheet with domain deformation
  - Optimal smoothing for reconstructing the sheets

QR code to the paper on arxiv

THANK YOU