Bootstrap for multistage sampling and without replacement sampling at the first stage

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Multistage sampling

Principle of multistage sampling

The population U of individuals is partitioned into M big units called **Primary Sampling Units** (PSUs); the small units in U are called the **Secondary Sampling Units** (SSUs).

- First stage: a sample S_I of PSUs is selected.
- Second stage: a sample of SSUs is drawn in the selected PSUs u_i.

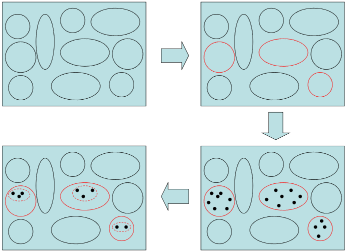
Multistage sampling consists in three stages of sampling, or more. In case of household surveys, a customary sampling design consists in

- selecting a sample of municipalities (PSUs),
- selecting a sample of districts inside the selected municipalities (SSUs),
- selecting a sample of households inside the selected districts (TSUs).

Motivation

Multistage sampling is mainly used for practical purpose:

- **Reducing the survey costs** when direct sampling would lead to a scattered sample. Using several stages of sampling enables to group the selected units.
- **Building of the sampling frame**. We only need a list of the final units inside the selected PSUs.



Examples

1. Household surveys: selection of a sample of municipalities (PSUs), of districts (SSUS) within, and of households (TSUs) inside (e.g., Ardilly, 2006).
2. Epidemiologic surveys: estimation of lead contamination by the selection of a sample of hospitals (PSUs), and then of children (SSUs) whose dwellings were investigated (Lucas, 2013).
3. PISA survey: in France, selection of a sample of schools (PSUs), and of a sample of students aged 15 within (SSUs).

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Framework

We consider a finite population $U = \{1, \ldots, N\}$ of N sampling units. The units are grouped inside N_I non-overlapping subpopulations $u_1, \ldots, u_{N_I}$ called primary sampling units (PSUs). We are interested in estimating the population total

$$Y = \sum_{k \in U} y_k = \sum_{u_i \in U_I} Y_i \quad \text{with} \quad Y_i = \sum_{k \in u_i} y_k,$$

for some variable of interest y.

We denote by:

- $\hat{Y}_i$ an unbiased estimator of Y_i, with design variance $V_i = V(\hat{Y}_i)$,
- $\hat{V}_i$ an unbiased estimator of V_i.

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Framework

We consider the asymptotic framework of Isaki and Fuller (1982):

- The population U belongs to a nested sequence $\{U_t\}$ of finite populations with increasing sizes N_t.
- The vector of values $y_{Ut} = (y_{1t}, \ldots, y_{Nt})^\top$ belongs to a sequence $\{y_{Ut}\}$ of N_t-vectors.

The subscript "t" is suppressed in the sequel.

In the population $U_I = \{u_1, \ldots, u_{N_I}\}$ of PSUs:

- a first-stage sample S_I is selected according to some sampling design $p_I(\cdot)$,
- if $u_i \in S_I$, a second-stage sample S_i is selected in u_i by means of any sampling design (census, stratified sampling, multistage sampling, ...).

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Assumptions

We assume:

- **Invariance of the second-stage designs:** the second stage of sampling is independent of S_I,
- **Independence of the second-stage designs:** the second-stage designs are independent from one PSU to another, conditionally on S_I.

We will also make use of the following assumptions:

H1: $N_I \underset{t\to\infty}{\longrightarrow} \infty$ and $n_I \underset{t\to\infty}{\longrightarrow} \infty$.

H2: There exists a constant C_1 such that $N_I^{-1} \sum_{u_i \in U_I} E|\hat{Y}_i|^4 < C_1$.

H3: There exists a constant C_2 such that $N_I^{-1} \sum_{u_i \in U_I} E(\hat{V}_i^2) < C_2$.

With replacement sampling of PSUs

With replacement simple random sampling of PSUs

The first-stage sample S_I^{WR} is selected by means of simple random sampling with replacement (SIR). The Hansen-Hurwitz estimator is

$$\hat{Y}_{WR} = \frac{N_I}{n_I} \sum_{j=1}^{n_I} \hat{Y}_{(j)},$$

where

- S_I^{WR} is obtained in $j = 1, \ldots, n_I$ independent draws,
- at each draw, a PSU $u_{(j)}$ with associated estimator $X_j \equiv \hat{Y}_{(j)}$.

The variance of $\hat{Y}_{WR}$ and an unbiased variance estimator are

$$V\left(\hat{Y}_{WR}\right) = \frac{N_I^2}{n_I} \left\{ \frac{N_I - 1}{N_I} S^2_{Y,U_I} + \frac{1}{N_I} \sum_{u_i \in U_I} V_i \right\}$$

$$v_{WR}\left(\hat{Y}_{WR}\right) = \frac{N_I^2}{n_I} s_X^2 \text{ with } s_X^2 = \frac{1}{n_I - 1} \sum_{j=1}^{n_I} (X_j - \bar{X}_n)^2.$$

With replacement simple random sampling of PSUs

The simple form of the variance estimator is primarily due to the writing of $\hat{Y}_{WR}$ as a sum of independent random variables.

Under the assumptions:

H1: $N_I \underset{t\to\infty}{\longrightarrow} \infty$ and $n_I \underset{t\to\infty}{\longrightarrow} \infty$,

H2: there exists a constant C_1 such that $N_I^{-1} \sum_{u_i \in U_I} E|\hat{Y}_i|^4 < C_1$,

we have

$$E\left|\frac{n_I}{N_I^2}\left\{v_{WR}\left(\hat{Y}_{WR}\right) - V\left(\hat{Y}_{WR}\right)\right\}\right|^2 \underset{t\to\infty}{\longrightarrow} 0.$$

A variance estimator for further stages inside the selected PSUs is not needed.

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Bootstrap for SIR of PSUs

We consider the with-replacement Bootstrap (BWR) of PSUs described in Rao and Wu (1988). The resample $(X_1^*, \ldots, X_m^*)^\top$ is obtained by sampling m times independently in $(X_1, \ldots, X_{n_I})$. Let

$$\bar{X}_m^* = \frac{1}{m} \sum_{j=1}^{m} X_j^* \quad \text{and} \quad s_X^{*2} = \frac{1}{m-1} \sum_{j=1}^{m} (X_j^* - \bar{X}_m^*)^2 .$$

Assume that (H1)-(H2) hold, and that $m \underset{t \to \infty}{\longrightarrow} \infty$. Then (Bickel and Freedman, 1981) :

$$\frac{\sqrt{m}(\bar{X}_m^* - \bar{X})}{s_X^*} \underset{\mathcal{L}}{\longrightarrow} \mathcal{N}(0,1).$$

Using the BWR with $m = n_I - 1$ enables to match the unbiased variance estimator $v_{WR}\left(\hat{Y}_{WR}\right)$ when estimating the total Y.

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Without replacement sampling of PSUs

Without replacement simple random sampling of PSUs

The first-stage sample S_I is selected by means of simple random sampling without replacement (SI). The Horvitz-Thompson estimator is

$$\hat{Y} = \frac{N_I}{n_I} \sum_{j=1}^{n_I} \hat{Y}_{(j)},$$

where

- S_I is obtained in $j = 1, \ldots, n_I$ without-replacement draws,
- at each draw, a PSU $u_{(j)}$ with associated estimator $Z_j \equiv \hat{Y}_{(j)}$.

The variance of $\hat{Y}$ and an unbiased variance estimator are

$$V(\hat{Y}) = \frac{N_I^2}{n_I} \left\{ (1 - f_I) S^2_{Y,U_I} + \frac{1}{N_I} \sum_{u_i \in U_I} V_i \right\}$$

$$v(\hat{Y}) = \frac{N_I^2}{n_I} \left\{ (1 - f_I) s^2_Z + \frac{1}{N_I} \sum_{u_i \in S_I} \hat{V}_i \right\} \text{ with } f_I = n_I / N_I.$$

Without replacement simple random sampling of PSUs

Since $\hat{Y}$ is a sum of dependent random variables, there is no such simple unbiased variance estimator as for SIR sampling of PSUs.

Under the assumptions:

H1: $N_I \underset{t\to\infty}{\longrightarrow} \infty$ and $n_I \underset{t\to\infty}{\longrightarrow} \infty$,

H2: there exists a constant C_1 such that $N_I^{-1} \sum_{u_i \in U_I} E|\hat{Y}_i|^4 < C_1$,

H3: There exists a constant C_2 such that $N_I^{-1} \sum_{u_i \in U_I} E(\hat{V}_i^2) < C_2$.

we have

$$E\left|\frac{n_I}{N_I^2}\left\{v(\hat{Y}) - V(\hat{Y})\right\}\right|^2 \underset{t\to\infty}{\longrightarrow} 0.$$

A variance estimator for further stages inside the PSUs is needed.

A coupling procedure between SI/SIR sampling of PSUs

Motivation

We would like to prove that, when the first-stage sampling fraction f_I is small:

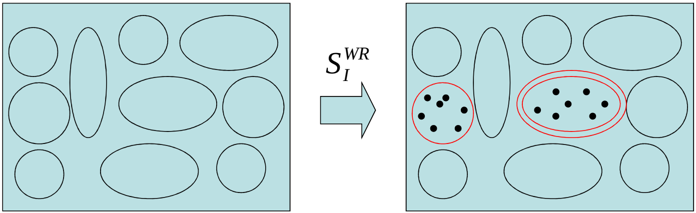
- the simplified variance estimator $v_{WR}(\hat{Y}) = \frac{N_I^2}{n_I} s_Z^2$ is also consistent in case of SI sampling of PSUs,
- the BWR of PSUs is suitable for SI sampling of PSUs.

We propose a coupling method (Hajek, 1960; Thorisson, 1980) to select jointly a with/without replacement sample of PSUs, in such a way that:

- $\bar{X}_n \simeq \bar{Z}_n$ and $s_X^2 \simeq s_Z^2$,
- $\frac{\sqrt{m}(\bar{X}_m^* - \bar{X})}{s_X^*} \simeq \frac{\sqrt{m}(\bar{Z}_m^* - \bar{Z})}{s_Z^*}$.

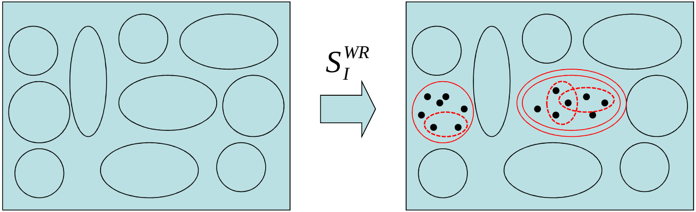
The coupling procedure

Step 1: draw S_I^{WR}. Denote by S_I^d the set of distinct PSUs in S_I^{WR}.



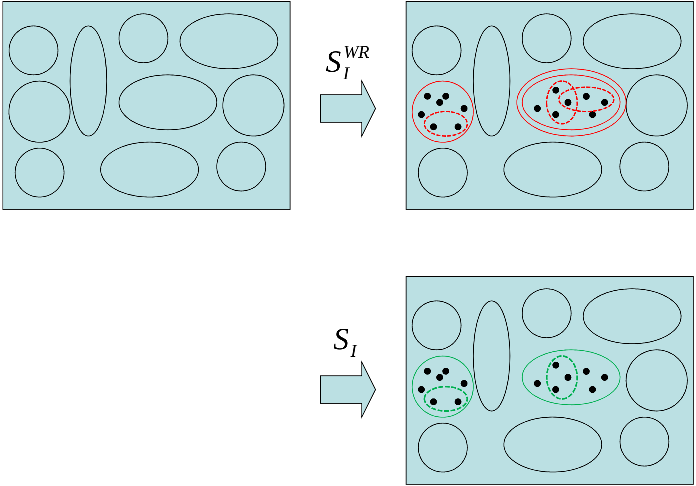
The coupling procedure

Step 2: each time $u_i \in S_I^{WR}$, select a second-stage sample $S_{i[j]}$.



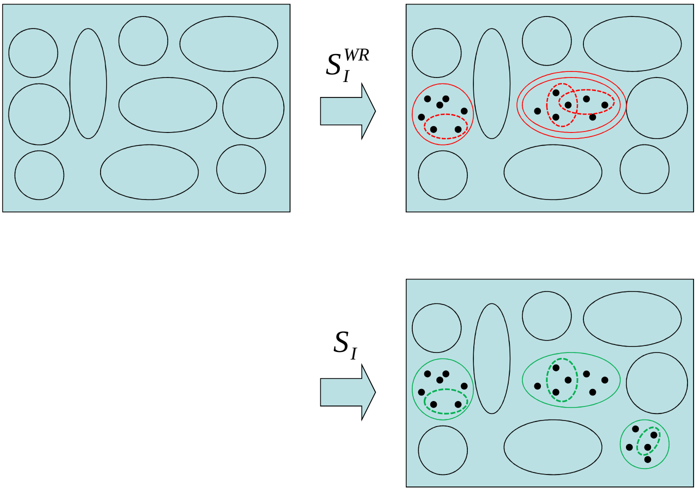
The coupling procedure

Step 3: initialize S_I with S_I^d, and $S_i = S_{i[1]}$ for $u_i \in S_I^d$.



The coupling procedure

Step 4: draw a complementary sample S_I^c, and S_i for $u_i \in S_I^c$.



The coupling procedure

Suppose that the samples S_I^{WR} and S_I are selected according to the coupling procedure. Then

$$\frac{E(\hat{Y}_{WR}-\hat{Y})^2}{V(\hat{Y}_{WR})} \leq \frac{n_I-1}{N_I-1}\left(\leq \frac{n_I}{N_I}\right). \quad (1)$$

Suppose that (H1)-(H2) hold, and that $f_I \underset{t\to\infty}{\longrightarrow} 0$. Then

$$E(\bar{Z}-\bar{X})^2 = o(n_I^{-1}) \quad \text{and} \quad \frac{V(\bar{Z})}{V(\bar{X})} \underset{t\to\infty}{\longrightarrow} 1.$$

Also, the simplified variance estimator $v_{WR}(\hat{Y}) = \frac{N_I^2}{n_I} s_Z^2$ is such that:

$$E\left|\frac{n_I}{N_I^2}\left\{v_{WR}(\hat{Y}) - v_{WR}(\hat{Y}_{WR})\right\}\right| \underset{t\to\infty}{\longrightarrow} 0.$$

With-replacement Bootstrap

We consider the same BWR of PSUs. Denote by

$$(Z_1^*, \ldots, Z_m^*)^\top$$

the resample obtained by sampling m times independently in $(Z_1, \ldots, Z_{n_I})$.

Let

$$\bar{Z}_m^* = \frac{1}{m} \sum_{j=1}^{m} Z_j^* \quad \text{and} \quad s_Z^{*2} = \frac{1}{m-1} \sum_{j=1}^{m} \left(Z_j^* - \bar{Z}_m^*\right)^2$$

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With-replacement Bootstrap

Mallows (1972) metric: let $1 \leq q < \infty$ and $d_q(\alpha,\beta) = \inf \{E\|X - Z\|^q\}^{1/q}$, where the infimum is taken over all couples (X,Z) with marginal distributions α and β.

Suppose that (H1) and (H2) hold, and that $m \underset{t\to\infty}{\longrightarrow} \infty$. Then :

$$d_2\left[\sqrt{m}(\bar{Z}_m^* - \bar{Z}), \sqrt{m}(\bar{X}_m^* - \bar{X})\right] \underset{t\to\infty}{\longrightarrow} 0, \quad (2)$$

$$d_1\left[s_Z^{*2}, s_X^{*2}\right] \underset{t\to\infty}{\longrightarrow} 0, \quad (3)$$

$$\frac{\sqrt{m}(\bar{Z}_m^* - \bar{Z})}{s_Z^*} \underset{\mathcal{L}}{\longrightarrow} \mathcal{N}(0,1). \quad (4)$$

Using the BWR with $m = n_I - 1$ enables to match the simplified variance estimator $v_{WR}\left(\hat{Y}\right)$ when estimating the total Y.

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Variance estimation

Suppose that y_k is a q-vector of interest. We are interested in a parameter

$$\theta = f(\mu_Y) \quad \text{with} \quad \mu_Y = N_I^{-1} \sum_{u_i \in U_I} Y_i,$$

where $f : \mathbb{R}^q \longrightarrow \mathbb{R}$ is differentiable with bounded partial derivatives and $f'(\mu_Y) \neq 0$. The plug-in estimator of θ is:

- $\hat{\theta} = f(\bar{Z})$ under SI sampling of PSUs,
- $\hat{\theta}_{WR} = f(\bar{X})$ under SIR sampling of PSUs.

Suppose that S_I^{WR} and S_I are selected according to the coupling procedure + assumptions (H1)-(H2) hold + $f_I \underset{t\to\infty}{\longrightarrow} 0$. Then :

$$E(\|\bar{Z} - \bar{X}\|^2) = o(n_I^{-1}),$$
$$E(\hat{\theta} - \hat{\theta}_{WR})^2 = o(n_I^{-1}).$$

with $\|\cdot\|$ the Euclidean norm.

Variance estimation

Suppose that the samples S_I^{WR} and S_I are selected according to the coupling procedure. Suppose that assumptions (H1)-(H2) hold, $f_I \underset{t\to\infty}{\longrightarrow} 0$ and $m \underset{t\to\infty}{\longrightarrow} \infty$. Then :

$$E(\|\bar{Z}^* - \bar{X}^*\|^2) = o(m^{-1}) + o(n_I^{-1}), \quad (5)$$

$$E(\hat{\theta}^* - \hat{\theta}_{WR}^*)^2 = o(m^{-1}) + o(n_I^{-1}). \quad (6)$$

This implies that

$$\frac{V(\hat{\theta}^*|Z_i)}{V(\hat{\theta}_{WR}^*|X_i)} \underset{Pr}{\longrightarrow} 1. \quad (7)$$

If the with-replacement Bootstrap provides consistent variance estimation for $\hat{\theta}_{WR}$, it is also consistent for $\hat{\theta}$.

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A simulation study

Simulation study

We generated 2 finite populations, each with $N_I = 2,000$ PSUs, so that the CV for the sizes N_i of PSUs was equal to 0 and 0.03. In each population, we generated for any PSU u_i:

$$\lambda_i = \lambda + \sigma\, v_i \qquad (8)$$

where the v_i's were generated according to a standardized normal distribution. For each SSU $k \in u_i$, we generated a couple of values according to the model

$$y_{1k} = \lambda_i + \{\rho^{-1}(1-\rho)\}^{0.5}\sigma\,(\alpha\,\epsilon_k + \eta_k), \qquad (9)$$
$$y_{2k} = \lambda_i + \{\rho^{-1}(1-\rho)\}^{0.5}\sigma\,(\alpha\,\epsilon_k + \nu_k), \qquad (10)$$

so as to have

- a coefficient of correlation approximately equal to 0.60,
- an intra-cluster correlation coefficient equal to 0.1 (similar results for 0.2 and 0.3).

Simulation study

From each population, we selected $B = 1,000$ two-stage samples by:

- SI sampling of size $n_I = 20, 40, 100$ or 200 at the first stage,
- systematic sampling of size $n_0 = 5$ or 10 at the second stage.

We want to estimate the variance of the substitution estimator for the parameters

$$
\begin{aligned}
R &= \frac{\mu_{y1}}{\mu_{y2}} \\
r &= \frac{\sum_{k\in U}(y_{1k}-\mu_{y1})(y_{2k}-\mu_{y2})}{\sqrt{\sum_{k\in U}(y_{1k}-\mu_{y1})^2}\sqrt{\sum_{k\in U}(y_{2k}-\mu_{y2})^2}},
\end{aligned}
$$

by using the BWR of PSUs. The true variance was approximated from a separate simulation run of $C = 20,000$ samples.

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Estimation of the ratio

			RB	RS	L	U	L+U
Pop. 1	$n_0 = 5$	$n_I = 20$	0.02	0.34	3.6	2.9	6.5
		$n_I = 40$	0.02	0.24	2.8	3.3	6.1
		$n_I = 100$	0.01	0.15	2.8	2.2	5.0
		$n_I = 200$	0.01	0.11	3.0	3.0	6.0
	$n_0 = 10$	$n_I = 20$	0.00	0.33	3.9	3.1	7.0
		$n_I = 40$	0.03	0.24	3.2	2.8	6.0
		$n_I = 100$	0.00	0.16	3.3	2.4	5.7
		$n_I = 200$	0.04	0.12	2.3	2.7	5.0
Pop. 2	$n_0 = 5$	$n_I = 20$	0.00	0.34	3.8	3.6	7.4
		$n_I = 40$	0.00	0.22	2.1	3.0	5.1
		$n_I = 100$	0.00	0.15	2.5	2.5	5.0
		$n_I = 200$	0.02	0.11	3.4	2.9	6.3
	$n_0 = 10$	$n_I = 20$	-0.01	0.33	3.7	2.6	6.3
		$n_I = 40$	0.00	0.24	3.2	3.5	6.7
		$n_I = 100$	0.02	0.16	3.3	2.2	5.5
		$n_I = 200$	0.02	0.11	2.6	2.6	5.2

Estimation of the coefficient of correlation

			RB	RS	L	U	L+U
Pop. 1	$n_0 = 5$	$n_I = 20$	0.01	0.41	3.8	3.2	7.0
		$n_I = 40$	0.00	0.29	2.9	2.8	5.7
		$n_I = 100$	0.02	0.19	3.2	2.6	5.8
		$n_I = 200$	0.01	0.14	2.8	2.1	4.9
	$n_0 = 10$	$n_I = 20$	-0.01	0.37	3.3	3.2	6.5
		$n_I = 40$	0.01	0.27	2.5	3.0	5.5
		$n_I = 100$	0.05	0.19	2.0	2.6	4.6
		$n_I = 200$	0.03	0.13	2.2	2.4	4.6
Pop. 2	$n_0 = 5$	$n_I = 20$	-0.01	0.41	4.2	3.2	7.4
		$n_I = 40$	0.02	0.31	2.6	2.9	5.5
		$n_I = 100$	0.02	0.19	3.0	2.9	5.9
		$n_I = 200$	0.01	0.14	2.2	2.7	4.9
	$n_0 = 10$	$n_I = 20$	0.01	0.40	2.9	3.7	6.6
		$n_I = 40$	0.00	0.28	4.1	2.8	6.9
		$n_I = 100$	0.02	0.17	2.9	2.4	5.3
		$n_I = 200$	0.04	0.13	2.5	3.4	5.9

References

- Ardilly, P. (2006). Les techniques de sondage. Editions Technip.
- Bickel, P.J., and Freedman, D.A. (1981). Some asymptotic theory for the Bootstrap. The Annals of Statistics, 9, 1196-1217.
- Hajek, J. (1960). Limiting distributions in simple random sampling from a finite population. Publications of the Mathematics Institute of the Hungarian Academy of Science, 5, 361-74.
- Isaki, C.T., and Fuller, W.A. (1982). Survey design under the regression superpopulation model. JASA, 77, 89-96.
- Lucas, J-P. (2013). Contamination des logements par le plomb : prévalence des logements à risque et identification des déterminants de la contamination. PhD dissertation, Université de Nantes.
- Mallows, C. (1972). A note on asymptotic joint normality. The Annals of Mathematical Statistics, 43, 508-515.
- Rao, J.N.K., Wu, C.F.J. (1988). Resampling inference with complex survey data. JASA, 83(401), 231-241.
- Särndal, C.-E., Swensson, B. and Wretman, J.H. (1992). Model Assisted Survey Sampling. New-York, Springer-Verlag.
- Thorisson, H. (2000). Coupling, stationarity, and regeneration. New York, Springer.

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